MONGE-AMPÈRE MEASURES ON PLURIPOLAR SETS

PER ÅHAG, URBAN CEGRELL, RAFAŁ CZYŻ, AND PHAM HOANG HIEP

Abstract. In this article we solve the complex Monge-Ampère equation for measures with large singular part. This result generalizes classical results by Demailly, Lelong and Lempert a.o., who considered singular parts carried on discrete sets. By using our result we obtain a generalization of Kołodziej's subsolution theorem. More precisely, we prove that if a non-negative Borel measure is dominated by a complex Monge-Ampère measure, then it is a complex Monge-Ampère measure.

1. Introduction

In this article we study the complex Monge-Ampère equation $(dd^cu)^n = \mu$, where μ is a given non-negative Radon measure and $(dd^c\cdot)^n$ denotes the complex Monge-Ampère operator. If μ puts mass on a pluripolar set, then the solution to $(dd^cu)^n = \mu$ cannot generally be uniquely determined (see, e.g. [13, 30]). Therefore the question of existence of solutions is our main interest. The first result was due to Lempert who, in [20, 21], obtained a positive result for the case when the support of the given measure is a single point. He considered solutions with real-analytic boundary values and logarithmic singularity near the support of the measure. The underlying domain was assumed to be a strictly convex domain in $\mathbb{C}^n$ (see also [4] and Theorem 1.5 in [15]). In this context, it is worth to mention the article [10], where Celik and Poletsky also studies the Monge-Ampère equation with the Dirac measure as given measure.

Throughout this article it is always assumed that Ω is a bounded hyperconvex domain (see section 2 for the definition of hyperconvex domain). Demailly proved (Theorem 4.3 in [11]) that $(dd^cg_{A_1})^n = (2\pi)^n\delta_z$ on a hyperconvex domain Ω, where δ_z is the Dirac measure at z, and g_z is the pluricomplex Green function (introduced in [16, 29]) with pole set containing a single point $A_1 = \{z\}$. In [19], Lelong introduced the pluricomplex Green function with a finite pole set, $A_k = \{z_1, \ldots, z_k\}$, and with positive weights $v_1, \ldots, v_k$, $v_l > 0$, $l = 1, \ldots, k$, and proved that $(dd^cg_{A_k})^n = (2\pi)^n \sum_{j=1}^k v_j^n\delta_{z_j}$ (Proposition 8 in [19]). The pluricomplex Green function is not a solution to the complex Monge-Ampère equation if we want the solution to have other boundary values than those which are identically zero. Given a discrete measure with compact support in a hyperconvex domain Ω, Zeriahi proved in [30] that the complex Monge-Ampère equation is solvable for certain continuous boundary values. In [27], Xing generalized Zeriahi's result in the case where the given boundary values are identically zero. Xing considered

2000 *Mathematics Subject Classification.* Primary 32W20; Secondary 32U15.
Key words and phrases. Complex Monge-Ampère operator, Dirichlet problem, pluripolar set, plurisubharmonic function.
The third-named author was partially supported by ministerial grant number N N201 3679 33.

measures that were majorized by the sum of a linear combination of countable numbers of Dirac measures with compact support and a certain regular Monge-Ampère measure.

We shall consider the class $\mathcal{E}$ introduced in [7]. It is the largest set of non-positive plurisubharmonic functions defined on a hyperconvex domain Ω for which the complex Monge-Ampère operator is well-defined (Theorem 4.5 in [7]). Let $u \in \mathcal{E}$ and $0 \leq g \leq 1$ be a $\chi_{\{u=-\infty\}}(dd^c u)^n$-measurable function that vanishes outside $\{u = -\infty\}$. We define

$$u^g = \inf_{\substack{f \in T \\ f \leq g}} \left(\sup\{u_\tau : f \leq \tau,\ \tau \text{ is a bounded lower semicontinuous function}\}\right)^* ,$$

where u_τ is as in Definition 4.2, T is the family of certain simple functions, and $(w)^*$ denotes the upper semicontinuous regularization of w. We prove that $u^g \in \mathcal{E}$ and $(dd^c u^g)^n = g(dd^c u)^n$. In particular, this implies that for any pluripolar Borel set E in Ω we have that $(dd^c u^{\chi_E})^n = \chi_E(dd^c u)^n$, where χ_E is the characteristic function for the set E in Ω (Theorem 4.7). Example 4.9 shows that our given singular measure $\chi_E(dd^c u)^n$ is not necessarily a discrete measure. Hence, Theorem 4.7 yields solutions to the complex Monge-Ampère equation for a larger class of singular measures than Zeriahi and Xing. The following statement, which is included in Theorem 4.13, is the main result of this article:

Theorem 4.13 (3): *Assume that μ is a non-negative Radon measure. If there exists a function $w \in \mathcal{E}$ such that $\mu \leq (dd^c w)^n$, then there exists a function $u \in \mathcal{E}$ such that $w + H \leq u \leq H$ and $(dd^c u)^n = \mu$.*

Theorem 4.13 is a generalization of the celebrated subsolution theorem by Kołodziej ([17]; for an alternative proof see section 4 in [18]). Example 5.4 in [8] shows that there exists a non-negative Radon measure μ such that there does not exist any function $u \in \mathcal{E}$ that satisfies $(dd^c u)^n = \mu$.

This article is organized as follows. In Section 2 some definitions will be recalled. One of the most powerful tools when working with the complex Monge-Ampère operator is the *comparison principle*. In Section 3 we obtain the comparison principle for certain functions in $\mathcal{E}$ (Corollary 3.2). To prove the comparison principle we shall follow an idea from [26] and firstly prove a Xing type inequality (Theorem 3.1). The last section is devoted to the proof of Theorem 4.13.

The authors would like to thank Egmont Porten and Alexander Rashkovskii for many valuable comments on and suggestions for this manuscript. This research was partly done during the third- and fourth-named authors' visit to Mid Sweden University in Sundsvall, Sweden, in 2006 and 2007. This article was completed during their visit to Umeå University in Umeå, Sweden, in 2008. They wish to thank the members of both Departments of Mathematics for their kind hospitality.

2. Background and definitions

Throughout this article it is always assumed that Ω is a bounded hyperconvex domain. Recall that $\Omega \subseteq \mathbb{C}^n$, $n \geq 1$ is a bounded hyperconvex domain if it is a bounded, connected, and open set, such that there exists a bounded plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$.

In this article we adapt the notation that $\mathcal{PSH}(\Omega)$ is the family of plurisubharmonic functions defined on Ω and $\mathcal{MPSH}(\Omega)$ for the maximal plurisubharmonic functions. For the definitions and basic facts of these functions we refer to [16].

We say that a bounded plurisubharmonic function φ defined on Ω belongs to $\mathcal{E}_0$ if $\lim_{z\to\xi}\varphi(z)=0$, for every $\xi\in\partial\Omega$, and $\int_\Omega (dd^c\varphi)^n<+\infty$. It was proved in Lemma 3.1 in [7] that $C_0^\infty(\Omega)\subset\mathcal{E}_0\cap C(\bar\Omega)-\mathcal{E}_0\cap C(\bar\Omega)$.

Definition 2.1. Let $\mathcal{E}$ $(=\mathcal{E}(\Omega))$ be the class of plurisubharmonic functions φ defined on Ω, such that for each $z_0\in\Omega$ there exists a neighborhood ω of z_0 in Ω and a decreasing sequence $[\varphi_j]_{j=1}^\infty$, $\varphi_j\in\mathcal{E}_0$, that converges pointwise to φ on ω as $j\to+\infty$, and

$$\sup_j\int_\Omega (dd^c\varphi_j)^n<+\infty\,.$$

Furthermore, let $\mathcal{F}$ $(=\mathcal{F}(\Omega))$ be the subset of $\mathcal{E}$ containing those functions with smallest maximal plurisubharmonic majorant identically zero and with finite total Monge-Ampère mass.

If there can be no misinterpretation a sequence $[\,\cdot\,]_{j=1}^\infty$ will be denoted by $[\,\cdot\,]$. Shiffman and Taylor gave an example in [25] that shows that it is not possible to extend the complex Monge-Ampère operator in a meaningful way to the whole class of plurisubharmonic functions and still have the range contained in the class of non-negative measures (see also [14]). In [7] the second-named author proved that the complex Monge-Ampère operator is well-defined on $\mathcal{E}$. As mentioned in the introduction he proved that $\mathcal{E}$ is the natural domain of definition for the complex Monge-Ampère operator (Theorem 4.5 in [7]). In [2], Błocki proved that $\mathcal{E}=\{\varphi\in\mathcal{PSH}(\Omega)\cap W^{1,2}_{loc}(\Omega):\ \varphi\leq 0\}$ when $n=2$, and showed that this equality is not valid for $n\geq 3$. Later, in [3], he obtained a complete characterization of $\mathcal{E}$ for $n\geq 1$. Another characterization of $\mathcal{E}$ was proved in [9] in terms of the so-called φ-capacity.

In this article a *fundamental sequence* $[\Omega_j]$ is always an increasing sequence of strictly pseudoconvex subsets of Ω such that for every $j\in\mathbb{N}$ we have that, $\Omega_j\Subset\Omega_{j+1}$, and $\bigcup_{j=1}^\infty\Omega_j=\Omega$. Here $\Subset$ denotes that Ω_j is relatively compact in Ω_{j+1}.

Definition 2.2. Let $u\in\mathcal{PSH}(\Omega)$, $u\leq 0$, and let $[\Omega_j]$ be a fundamental sequence Ω_j . The function u^j is then defined by

$$u^j=\sup\left\{\varphi\in\mathcal{PSH}(\Omega):\varphi\leq u\ \text{ on }\ \mathcal{C}\Omega_j\right\},$$

where $\mathcal{C}\Omega_j$ denotes the complement of Ω_j in Ω.

Let $[\Omega_j]$ be a fundamental sequence and let $u\in\mathcal{PSH}(\Omega)$, $u\leq 0$, then $u^j\in\mathcal{PSH}(\Omega)$ and $u^j=u$ on $\mathcal{C}\Omega_j$. Definition 2.2 implies that $[u^j]$ is an increasing sequence and therefore $\lim_{j\to+\infty}u^j$ exists q.e. (quasi-everywhere) on Ω. Hence, the function $\tilde u$ defined by $\tilde u=\left(\lim_{j\to+\infty}u^j\right)^*$ is plurisubharmonic on Ω. Moreover, if $u\in\mathcal{E}$, then by [7] we have that $\tilde u\in\mathcal{E}$, since $u\leq\tilde u\leq 0$, and by [2, 3] it follows that $\tilde u$ is maximal on Ω. Let $u,v\in\mathcal{E}$ and $\alpha\in\mathbb{R}$, $\alpha\geq 0$, then it follows from Definition 2.2 that $\widetilde{u+v}\geq\tilde u+\tilde v$ and $\widetilde{\alpha u}=\alpha\,\tilde u$. Moreover, if $u\geq v$, then $\tilde u\geq\tilde v$. It follows from [2, 3] that $\mathcal{E}\cap\mathcal{MPSH}(\Omega)=\{u\in\mathcal{E}:\tilde u=u\}$. Set

$$\mathcal{N}=\{u\in\mathcal{E}:\tilde u=0\}\,.$$

Then we have that $\mathcal{N}$ is a convex cone and that $\mathcal{N}$ is precisely the set of functions in $\mathcal{E}$ with smallest maximal plurisubharmonic majorant identically zero.

Definition 2.3. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{N}\}$. We say that a plurisubharmonic function u defined on Ω belongs to the class $\mathcal{K}(\Omega, H)$, $H \in \mathcal{E}$, if there exists a function $\varphi \in \mathcal{K}$ such that

$$H \geq u \geq \varphi + H\,.$$

Note that $\mathcal{K}(\Omega, 0) = \mathcal{K}$. The following approximation theorem was proved by the second-named author in [7].

Theorem 2.4. *Let $u \in \mathcal{PSH}(\Omega)$, $u \leq 0$. Then there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$, which converges pointwise to u on Ω, as j tends to $+\infty$.*

Theorem 2.4 yields among other things the following simple and useful observation.

Proposition 2.5. *Let $H \in \mathcal{E}$ and $u \in \mathcal{PSH}(\Omega)$ be such that $u \leq H$, then there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0(H)$, that converges pointwise to u on Ω, as j tends to $+\infty$. Moreover, if $H \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$, then the decreasing sequence $[u_j]$ can be chosen such that $u_j \in \mathcal{E}_0(H) \cap C(\bar{\Omega})$.*

Proof. Theorem 2.4 implies that there exists a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$, that converges pointwise to u, as $j \to +\infty$. If $v_j = \max(u, \varphi_j + H)$, then $[v_j]$, $v_j \in \mathcal{E}_0(H)$, is a decreasing sequence that converges pointwise to u, as $j \to +\infty$, and the first statement is completed.

For the second statement assume that $H \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ and let $\varphi \in \mathcal{E}_0 \cap C(\bar{\Omega})$, not identically 0. Choose a fundamental sequence, $[\Omega_j]$ of Ω such that for each $j \in \mathbb{N}$ we have that $\varphi \geq -\frac{1}{2j^2}$ on $\mathcal{C}\Omega_j$. Let $[v_j]$, $v_j \in \mathcal{PSH}(\Omega_j) \cap C^\infty(\Omega_j)$, be a decreasing sequence that converges pointwise to u, as $j \to +\infty$, and $v_j \leq H + \frac{1}{2j}$ on Ω_{j+1}. Set

$$u_j' = \begin{cases} \max\left(v_j - \frac{1}{j}\,, j\varphi + H\right) & \text{on } \Omega_j \\ j\varphi + H & \text{on } \mathcal{C}\Omega_j\,. \end{cases}$$

Then $[u_j']$, $u_j' \in \mathcal{E}_0(H) \cap C(\bar{\Omega})$, converges pointwise to u on Ω, as $j \to +\infty$, but $[u_j']$ is not necessarily decreasing. Let $u_j = \sup_{k \geq j} u_k'$. The construction of u_j' implies that

$$u_j' + \frac{1}{j} \geq u_{j+1}' + \frac{1}{j+1}$$

and therefore for each $j \in \mathbb{N}$ fixed it follows that

$$\left[\max\left(u_j', u_{j+1}', \ldots, u_{m-1}', u_m' + \frac{1}{m}\right)\right]_{m=j}^{\infty}$$

decreases pointwise on Ω to u_j, as $m \to +\infty$. Thus, u_j is an upper semicontinuous function and we have that $u_j \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. Moreover $[u_j]$ is decreasing and converges pointwise to u on Ω, as $j \to +\infty$. □

Remark. If H is unbounded, then each function u_j is necessarily unbounded.

3. Some auxiliary results

Theorem 3.1. *Let $H \in \mathcal{E}$. If $u \in \mathcal{N}(H)$ and $v \in \mathcal{E}$ is such that $v \leq H$ on Ω, then for all $w_j \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $-1 \leq w_j \leq 0$, $j = 1, 2, ..., n$, we have the following*

inequality:

$$\frac{1}{n!}\int_{\{u<v\}}(v-u)^n dd^c w_1\wedge\cdots\wedge dd^c w_n+\int_{\{u<v\}}(-w_1)(dd^c v)^n\leq$$
$$\leq\int_{\{u<v\}}(-w_1)(dd^c u)^n+\int_{\{u=v=-\infty\}}(-w_1)(dd^c u)^n\,. \quad (3.1)$$

Proof. Let $u\in\mathcal{N}(H)$, i.e., $u\in\mathcal{PSH}(\Omega)$ and there exists a function $\varphi\in\mathcal{N}$ such that

$$H\geq u\geq\varphi+H\,.$$

Let $[\Omega_j]$ be a fundamental sequence in Ω and let φ^j be defined as in Definition 2.2. The assumption that $v\leq H$ implies that for $\varepsilon>0$ the following inequality holds

$$u\geq\varphi+H=\varphi^j+H\geq\varphi^j+v-\varepsilon\quad\text{on }\mathcal{C}\Omega_j\,.$$

Theorem 4.9 in [22] implies that

$$\frac{1}{n!}\int_{\{u<v-\varepsilon+\varphi^j\}}(v-\varepsilon+\varphi^j-u)^n dd^c w_1\wedge\cdots\wedge dd^c w_n+\int_{\{u<v-\varepsilon+\varphi^j\}}(-w_1)(dd^c v)^n\leq$$
$$\leq\int_{\{u\leq v-\varepsilon\}}(-w_1)(dd^c u)^n\,.$$

We have that

$$[\chi_{\{u<v-\varepsilon+\varphi^j\}}(v-\varepsilon+\varphi^j-u)^n]_{j=1}^{\infty}\quad\text{and}\quad[\chi_{\{u<v-\varepsilon+\varphi^j\}}]_{j=1}^{\infty} \quad (3.2)$$

are two increasing sequences of functions that converges q.e. on Ω to $\chi_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^n$ and $\chi_{\{u<v-\varepsilon\}}$, respectively, as $j\to+\infty$. Theorem 5.11 in [7] implies that $dd^c w_1\wedge\cdots\wedge dd^c w_n\ll C_n$ and $\chi_{\{v>-\infty\}}(dd^c v)^n\ll C_n$. Here C_n denotes the usual C_n-capacity and $\mu\ll C_n$ denotes that the measure μ is absolutely continuous with respect to C_n (see e.g. [18] for background). We therefore have that $[\chi_{\{u<v-\varepsilon+\varphi^j\}}(v-\varepsilon+\varphi^j-u)^n]_{j=1}^{\infty}$ converges to $\chi_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^n$ a.e. w.r.t. $dd^c w_1\wedge\cdots\wedge w_n$ and that $[\chi_{\{u<v-\varepsilon+\varphi^j\}}]_{j=1}^{\infty}$ converges to $\chi_{\{u<v-\varepsilon\}}$ a.e. w.r.t. $\chi_{\{v>-\infty\}}(dd^c v)^n$. The monotone convergence theorem yields that

$$\frac{1}{n!}\int_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^n dd^c w_1\wedge\cdots\wedge dd^c w_n+\int_{\{u<v-\varepsilon\}}(-w_1)(dd^c v)^n\leq$$
$$\leq\int_{\{u\leq v-\varepsilon\}}(-w_1)(dd^c u)^n\,.$$

Inequality (3.1) is now obtain by letting $\varepsilon\to 0^+$. □

Corollary 3.2. *Let $u,v,H\in\mathcal{E}$ be such that $(dd^c u)^n$ vanishes on all pluripolar sets in Ω and $(dd^c u)^n\leq(dd^c v)^n$. Consider the following two conditions*

(1) $\varliminf_{z\to\zeta}(u(z)-v(z))\geq 0$ *for every* $\zeta\in\partial\Omega$,
(2) $u\in\mathcal{N}(H)$, $v\leq H$.

If one of the above conditions is satisfied, then $u\geq v$ on Ω.

Proof. Assume that $u,v\in\mathcal{E}$ are such that $(dd^c u)^n$ vanishes on all pluripolar sets in Ω and $(dd^c u)^n\leq(dd^c v)^n$.

(1): Moreover, assume that

$$\varliminf_{z\to\zeta}(u(z)-v(z))\geq 0$$

for every $\zeta\in\partial\Omega$. Let $\varepsilon>0$. Theorem 4.9 in [22] implies that

$$\begin{aligned}
&\frac{\varepsilon^n}{n!}C_n(\{u+2\varepsilon<v\})\\
&\quad\leq \sup\left\{\frac{1}{n!}\int_{\{u+2\varepsilon<v\}}(v-u-2\varepsilon)^n(dd^cw)^n : w\in\mathcal{PSH}(\Omega),\ 0\leq w\leq 1\right\}\\
&\quad\leq \sup\left\{\frac{1}{n!}\int_{\{u+\varepsilon<v\}}(v-u-\varepsilon)^n(dd^cw)^n : w\in\mathcal{PSH}(\Omega),\ 0\leq w\leq 1\right\}\\
&\qquad\qquad\qquad\leq \frac{1}{n!}\int_{\{u+\varepsilon<v\}}(-w)[(dd^cu)^n-(dd^cv)^n]\leq 0\,. \qquad (3.3)
\end{aligned}$$

Thus, $u+2\varepsilon\geq v$. Let $\varepsilon\to 0^+$, then $u\geq v$ on Ω.

(2): In this case assume that $u\in\mathcal{N}(H)$ and $v\leq H$. Since $u\in\mathcal{N}(H)$ there exists a function $\varphi\in\mathcal{N}$ such that $H+\varphi\leq u\leq H$. Let φ^j be defined as in Definition 2.2 and let $\varepsilon>0$. Similarly as in (3.3) we get that $u+2\varepsilon\geq v+\varphi^j$. Let $\varepsilon\to 0^+$. Hence $u\geq v$ on Ω. □

Remark. In Corollary 3.2, the assumption that $(dd^cu)^n$ vanishes on all pluripolar sets is essential.

Lemma 3.3. *Let $u,v\in\mathcal{N}(H)$, be such that $u\leq v$ and $\int_\Omega(-\varphi)dd^cu\wedge T<+\infty$, $\varphi\in\mathcal{PSH}(\Omega)$, $\varphi\leq 0$. Then the following inequality holds*

$$\int_\Omega(-\varphi)dd^cu\wedge T\geq\int_\Omega(-\varphi)dd^cv\wedge T\,, \qquad (3.4)$$

where $T=dd^cw_2\wedge\cdots\wedge dd^cw_n$, $w_2,\dots,w_n\in\mathcal{E}$.

Proof. Let $[\Omega_s]$ be a fundamental sequence in Ω. By the assumption that $u\in\mathcal{N}(H)$ there exists a function $\psi\in\mathcal{N}$ such that $H\geq u\geq\psi+H$. For each $j\in\mathbb{N}$ consider the function defined by $v_j=\max(u,\psi^j+v)$, where ψ^j is defined as in Definition 2.2. This construction imply that $v_j\in\mathcal{E}$, $v_j=u$ on $\mathcal{C}\Omega_j$, $u\leq v_j$, and $[v_j]$ is an increasing sequence that converges pointwise to v q.e. on Ω, as $j\to+\infty$. Theorem 2.4 implies that there exists a decreasing sequence $[\varphi_k]$, $\varphi_k\in\mathcal{E}_0\cap C(\bar\Omega)$, that converges pointwise to φ, as $j\to+\infty$. We have by Stokes' theorem that for each $s\geq j$ it holds that

$$\int_{\Omega_s}(-\varphi_k)dd^cu\wedge T-\int_{\Omega_s}(-\varphi_k)dd^cv_j\wedge T=\int_{\Omega_s}(v_j-u)dd^c\varphi_k\wedge T\geq 0\,.$$

By letting $s\to+\infty$ we get that

$$\int_\Omega(-\varphi_k)dd^cu\wedge T\geq\int_\Omega(-\varphi_k)dd^cv_j\wedge T\,. \qquad (3.5)$$

The function φ_k is bounded and therefore it follows from [7, remark on pg. 175] that $(-\varphi_k)dd^cv_j\wedge T$ converges to $(-\varphi_k)dd^cv\wedge T$ in the weak*-topology, as $j\to+\infty$, which yields that

$$\lim_{j\to+\infty}\int_\Omega(-\varphi_k)dd^cv_j\wedge T\geq\int_\Omega(-\varphi_k)dd^cv\wedge T\,. \qquad (3.6)$$

Inequality (3.5) and (3.6) imply that inequality (3.4) holds for φ_k and the monotone convergence theorem completes this proof, when we let $k \to +\infty$. $\square$

Corollary 3.4. *Let $H \in \mathcal{E}$ and $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \leq 0$. If $[u_j]$, $u_j \in \mathcal{N}(H)$, is a decreasing sequence that converges pointwise on Ω to a function $u \in \mathcal{N}(H)$ as j tends to $+\infty$, then*

$$\lim_{j\to+\infty} \int_\Omega (-\varphi)(dd^c u_j)^n = \int_\Omega (-\varphi)(dd^c u)^n\,. \tag{3.7}$$

Proof. Let $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \leq 0$, and let $u_j, u \in \mathcal{N}(H)$ be such that $u \leq u_j$. If $\int_\Omega (-\varphi)(dd^c u)^n = +\infty$, then (3.7) follows immediately and therefore we can assume that $\int_\Omega (-\varphi)(dd^c u)^n < +\infty$. Lemma 3.3 implies that $[\int_\Omega (-\varphi)\,(dd^c u_j)^n]$ is an increasing sequence that is bounded above by $\int_\Omega (-\varphi)\,(dd^c u)^n$. From Corollary 5.2 in [7] it follows that the sequence $[(-\varphi)(dd^c u_j)^n]$ converges to $(-\varphi)(dd^c u)^n$ in the weak*-topology, as $j \to +\infty$, and the desired limit of the total masses is valid. $\square$

Lemma 3.5. *Let $H \in \mathcal{E}$ and let $u, v \in \mathcal{N}(H)$ be such that $u \leq v$. Then for all $w_j \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $-1 \leq w_j \leq 0$, $j = 1, 2, ..., n$, $\int_\Omega (-w_1)(dd^c u)^n < +\infty$, we have that the following inequality holds*

$$\frac{1}{n!}\int_\Omega (v-u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c v)^n \leq$$
$$\leq \int_\Omega (-w_1)(dd^c u)^n\,. \tag{3.8}$$

Proof. First we assume that $u, v \in \mathcal{E}_0(H)$. By definition there exists a function $\varphi \in \mathcal{E}_0$ such that $H \geq u \geq \varphi + H$. For each $\varepsilon > 0$ small enough choose $K \Subset \Omega$ such that $\varphi \geq -\varepsilon$ on $\mathcal{C}K$. Hence,

$$u \geq \varphi + H \geq -\epsilon + H \geq -\epsilon + v \ \text{ on } \ \mathcal{C}K\,,$$

and therefore it follows that $\max(u, v - \epsilon) = u$ on $\mathcal{C}K$. By using Proposition 3.1 in [22] we get that

$$\frac{1}{n!}\int_\Omega (\max(u, v-\varepsilon) - u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c \max(u, v-\varepsilon))^n$$
$$\leq \int_\Omega (-w_1)(dd^c u)^n\,.$$

By letting $\varepsilon \to 0^+$ we obtain inequality (3.8) in the case when $u, v \in \mathcal{E}_0(H)$. Using the case when $u, v \in \mathcal{E}_0(H)$ together with Proposition 2.5 and Corollary 3.4 we complete the proof. $\square$

An immediate consequence of Lemma 3.5 is the following identity principle. Theorem 3.6 play a technical prominent role in Section 4. In particular, this generalizes for example Lemma 6.3 in [24], Theorem 3.15 in [7], and the corresponding result in [22].

Theorem 3.6. *Let $H \in \mathcal{E}$. If $u, v \in \mathcal{N}(H)$ is such that $u \leq v$, $(dd^c u)^n = (dd^c v)^n$ and $\int_\Omega (-w)(dd^c u)^n < +\infty$ for some $w \in \mathcal{E}$ which is not identically 0, then $u = v$ on Ω.*

Theorem 3.7. *Assume that μ is a non-negative measure defined on Ω by $\mu = (dd^c\varphi)^n$, $\varphi \in \mathcal{N}$ with $\mu(A) = 0$ for every pluripolar set $A \subseteq \Omega$. Then for every $H \in \mathcal{E}$ such that $(dd^cH)^n \leq \mu$ there exists a uniquely determined function $u \in \mathcal{N}(H)$ such that $(dd^cu)^n = \mu$ on Ω.*

Proof. The uniqueness part of this theorem follows by the comparison principle in Corollary 3.2. We will proceed with the existence part. Theorem 2.4 implies that there exists a decreasing sequence $[H_k]$, $H_k \in \mathcal{E}_0 \cap C(\bar{\Omega})$, that converges pointwise to H, as $j \to +\infty$. Let $[\Omega_j]$ be a fundamental sequence in Ω. For each $j, k \in \mathbb{N}$ let H_k^j be the function defined as in Definition 2.2, i.e.,

$$H_k^j = \sup\left\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \leq H_k \text{ on } \mathcal{C}\Omega_j\right\},$$

Then $H_k^j \in \mathcal{E}_0(\Omega)$ and H_k^j is maximal on Ω_j. Consider the measure $\mu_j = \chi_{\Omega_j}\mu$ defined on Ω, where χ_{Ω_j} is the characteristic function for the set Ω_j in Ω . For each $j \in \mathbb{N}$ the measure μ_j is a compactly supported Borel measure defined on Ω, μ_j vanishes on all pluripolar sets in Ω and $\mu_j(\Omega_j) < \mu_j(\Omega) < +\infty$. Therefore it follows from Lemma 5.14 in [7] that there exists a uniquely determined function $\varphi_j \in \mathcal{F}(\Omega_j)$ such that $(dd^c\varphi_j)^n = \mu_j$ on Ω_j. Moreover, from Theorem 4.1 in [8] it follows that there exists functions $u_{j,k} \in \mathcal{F}(\Omega_j, H_k^j)$ such that $(dd^cu_{j,k})^n = \mu_j$ on Ω_j. Corollary 3.2 implies that

$$H_k^j \geq u_{j,k} \geq \varphi_j + H_k^j \quad \text{on} \quad \Omega_j\,, \tag{3.9}$$

since $(dd^cu_{j,k})^n \leq \left(dd^c(\varphi_j + H_k^j)\right)^n$ and H_k^j is maximal on Ω_j. The comparison principle (Corollary 3.2) yields that $[u_{j,k}]_{k=1}^{\infty}$ is a decreasing sequence. Let $k \to +\infty$ and set $u_j = \lim_{k\to+\infty} u_{j,k}$, then (3.9) gives us that $H^j \geq u_j \geq \varphi_j + H^j$ on Ω_j, i.e., $u_j \in \mathcal{F}(\Omega_j, H^j) \subseteq \mathcal{N}(\Omega_j, H^j)$. From the assumption that $\mu \geq (dd^cH)^n$ we get that $(dd^cu_j)^n = \mu_j = \chi_{\Omega_j}\mu = \mu \geq (dd^cH)^n$ on Ω_j and therefore it follows from Corollary 3.2 that $u_j \leq H$ on Ω_j. The construction of μ_j and the fact that $[\Omega_j]$ is an increasing sequence imply that $(dd^cu_j)^n = (dd^cu_{j+1})^n$ on Ω_j. Hence $[u_j]$ is decreasing and

$$H \geq u_j \geq \varphi + H \quad \text{on} \quad \Omega_j\,. \tag{3.10}$$

Thus, the function $u = (\lim_{j\to+\infty} u_j) \in \mathcal{N}(\Omega, H)$ is such that $(dd^cu)^n = \mu$ on Ω. $\square$

Remark. Let μ be a non-negative measure defined on Ω such that it vanishes on pluripolar subsets of Ω and that there exists a function $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi < 0$, such that $\int_\Omega(-\varphi)\, d\mu < +\infty$. Then it follows from [8] that there exists a uniquely determined function $\varphi \in \mathcal{N}$ such that $(dd^c\varphi)^n = \mu$.

4. Monge-Ampère measures carried on pluripolar sets

Lemma 4.1 is due to Demailly ([12]). Here we include his proof in our setting.

Lemma 4.1. *Let $u, u_k, v \in \mathcal{E}$, $k = 1, \ldots, n-1$, with $u \geq v$ on Ω and set $T = dd^cu_1 \wedge \cdots \wedge dd^cu_{n-1}$. Then*

$$\chi_{\{u=-\infty\}}dd^cu \wedge T \leq \chi_{\{v=-\infty\}}dd^cv \wedge T\,.$$

Proof. Let $\varepsilon > 0$. Set $w_j = \max((1-\varepsilon)u - j, v)$. Then $w_j = (1-\varepsilon)u - j$ on the open set $\{v < -\frac{j}{\varepsilon}\}$ and therefore we have that

$$dd^c w_j \wedge T = (1-\varepsilon) dd^c u \wedge T \text{ on } \left\{v < -\frac{j}{\varepsilon}\right\}.$$

Hence $dd^c w_j \wedge T \geq (1-\varepsilon)\chi_{\{u=-\infty\}} dd^c u \wedge T$. Let $j \to +\infty$, then

$$dd^c v \wedge T \geq (1-\varepsilon)\chi_{\{u=-\infty\}} dd^c u \wedge T \text{ on } \Omega.$$

The proof is completed as $\varepsilon \to 0^+$. □

Remark. For $j = 1, \dots, n$, let $u_j, v_j \in \mathcal{E}$, and $u_j \geq v_j$, then Lemma 4.1 implies that

$$\int_A dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq \int_A dd^c v_1 \wedge \cdots \wedge dd^c v_n,$$

for every pluripolar Borel set $A \subseteq \Omega$.

Remark. Let $u, v \in \mathcal{E}$ and assume that $(dd^c v)^n$ vanishes on pluripolar sets. If $u \geq v$, then it follows from Lemma 4.1 that $(dd^c u)^n$ vanishes on pluripolar sets.

Definition 4.2. Let $u \in \mathcal{E}$ and $0 \leq \tau$ be a bounded lower semicontinuous function. Then we define

$$u_\tau = \sup\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \leq \tau^{1/n} u\}.$$

Definition 4.2 yields the following elementary properties:

(1) If $u, v \in \mathcal{E}$ with $u \leq v$, then $u_\tau \leq v_\tau$.

(2) If $u \in \mathcal{E}$, then $0 \geq u_\tau \geq \|\tau\|_{L^\infty(\Omega)}^{1/n} u \in \mathcal{E}$. Hence, by [7] we have that $u_\tau \in \mathcal{E}$.

(3) If τ_1, τ_2 are bounded lower semicontinuous functions with $\tau_1 \leq \tau_2$, then $u_{\tau_1} \geq u_{\tau_2}$.

(4) If $u \in \mathcal{E}$, then $\operatorname{supp}(dd^c u_\tau)^n \subseteq \operatorname{supp}\tau$ and if $\operatorname{supp}\tau$ is compact then $u_\tau \in \mathcal{F}$.

(5) If $[\tau_j]$, $0 \leq \tau_j$ is an increasing sequence of bounded lower semicontinuous functions that converges pointwise to a bounded lower semicontinuous function τ, as j tends to $+\infty$, then $[u_{\tau_j}]$ is a decreasing sequence that converges pointwise to u_τ, as j tends to $+\infty$.

Lemma 4.3. *Let $u \in \mathcal{E}$ and let K be a compact pluripolar subset of Ω. Then*

$$(dd^c u_K)^n = \chi_K (dd^c u)^n,$$

where u_{χ_O} is as in Definition 4.2 and

$$u_K = \left(\sup\{u_{\chi_O} : K \subset O \subset \Omega,\ O \text{ is open}\}\right)^*.$$

Proof. Choose a decreasing sequence $[O_j]$, $O_j \subseteq \Omega$, such that $K = \bigcap_j O_j$. Then $[u_{\chi_{O_j}}]$ is an increases sequence that converges to u_K outside a pluripolar set, as $j \to +\infty$, and $\operatorname{supp}(dd^c u_K)^n \subseteq \bigcap \bar{O}_j = K$. We have that $u_{\chi_{O_j}} = u$ on O_j hence $\left(dd^c u_{\chi_{O_j}}\right)^n \geq \chi_K (dd^c u)^n$, so $(dd^c u_K)^n \geq \chi_K (dd^c u)^n$. On the other hand, $u_K \geq u$ and therefore we have by Lemma 4.1 that

$$\int_K (dd^c u_K)^n \leq \int_K (dd^c u)^n \text{ and } (dd^c u_K)^n = \chi_K (dd^c u)^n.$$

□

Lemma 4.4. *Let $u_1, \dots, u_n \in \mathcal{E}$. Then*

$$\int_A dd^c u_1 \wedge \dots \wedge dd^c u_n \leq \left(\int_A (dd^c u_1)^n\right)^{1/n} \cdots \left(\int_A (dd^c u_n)^n\right)^{1/n},$$

for every pluripolar Borel set $A \subset \Omega$.

Proof. Without loss of generality we can assume that A is a compact pluripolar set and $u_1, \dots, u_n \in \mathcal{F}$. Let $[G_j]$ be a decreasing sequence of open subsets of Ω with $\bigcap_j G_j = A$. Corollary 5.6 in [7] yields that

$$\int_\Omega dd^c u_{1_{G_j}} \wedge \dots \wedge dd^c u_{n_{G_j}} \leq \left(\int_\Omega \left(dd^c u_{1_{G_j}}\right)^n\right)^{1/n} \cdots \left(\int_\Omega \left(dd^c u_{n_{G_j}}\right)^n\right)^{1/n}$$

For $1 \leq k \leq n$ we have that $u_{k_{G_j}} = u_k$ on G_j and $\operatorname{supp}(dd^c u_{k_{G_j}})^n \subset \bar{G}_j \subset \bar{G}_1$, hence

$$\int_{G_j} dd^c u_1 \wedge \dots \wedge dd^c u_n \leq \left(\int_{\bar{G}_1} \left(dd^c u_{1_{G_j}}\right)^n\right)^{1/n} \cdots \left(\int_{\bar{G}_1} \left(dd^c u_{n_{G_j}}\right)^n\right)^{1/n}.$$

Let $j \to +\infty$. Lemma 4.3 then yields that

$$\int_A dd^c u_1 \wedge \dots \wedge dd^c u_n \leq \left(\int_{\bar{G}_1} (dd^c u_{1_A})^n\right)^{1/n} \cdots \left(\int_{\bar{G}_1} (dd^c u_{n_A})^n\right)^{1/n}.$$

□

For $u \in \mathcal{E}$ we write $\mu_u = \chi_{\{u=-\infty\}}(dd^c u)^n$ and define S to be the class of simple functions $f = \sum_{j=1}^m \alpha_j \chi_{E_j}$, $\alpha_j > 0$, where E_j are pairwise disjoint and μ-measurable such that f is compactly supported and vanishes outside $\{u = -\infty\}$. We write T for functions in S where the E_j's are compact.

Definition 4.5. Let $u \in \mathcal{E}$ and $0 \leq g \leq 1$ be a μ_u-measurable function. We define

$$u^g = \inf_{\substack{f \in T \\ f \leq g}} \left(\sup\{u_\tau : f \leq \tau,\ \tau \text{ is a bounded lower semicontinuous function}\}\right)^*.$$

From Definition 4.5 it follows that $u \leq u^g \leq 0$ and if $g_1 \leq g_2$, then $u^{g_1} \geq u^{g_2}$. Furthermore, if $g \in T$, then

$$u^g = \left(\sup\{u_\tau : g \leq \tau,\ \tau \text{ is a bounded lower semicontinuous function}\}\right)^* \in \mathcal{F}.$$

Lemma 4.6. *Let $u \in \mathcal{E}$ and $g \in S$, then $u^g \in \mathcal{F}$ and $(dd^c u^g)^n = g(dd^c u)^n$.*

Proof. Assume first that $g \in T$. Then $u^g \in \mathcal{F}$ as already noted. Let $g = \sum_{k=1}^m \alpha_k \chi_{A_k}$ and consider $u_k = u^{\alpha_k \chi_{A_k}}$. Then for $1 \leq k \leq m$ we have that $u_1 + \dots + u_m \leq u^g \leq u_k$ so if $B \subseteq \bigcup_{k=1}^m A_k$, then it follows from Lemma 4.1 that

$$\int_B (dd^c u_k)^n \leq \int_B (dd^c u^g)^n \leq \int_B (dd^c (u_1 + \dots + u_m))^n \quad 1 \leq k \leq m.$$

Hence, if $B \subset A_k$, then it follows from Lemma 4.3 that $\int_B (dd^c u_k)^n = \alpha_k \int_B (dd^c u)^n$ and from Lemma 4.4 we have that

$$\alpha_k \int_B (dd^c u)^n = \int_B (dd^c (u_1 + \dots + u_m))^n.$$

Hence,

$$\alpha_k \int_B (dd^c u_k)^n \leq \int_B (dd^c u^g)^n \leq \alpha_k \int_B (dd^c u)^n \quad 1 \leq k \leq m\,.$$

for all Borel sets $B \subset A_k$, $k = 1, ..., m$. Thus $(dd^c u^g)^n = g(dd^c u)^n$.

Assume now that $g \in S$, i.e., $g = \sum_{j=1}^m \alpha_j \chi_{E_j}$, $\alpha_j > 0$, E_j are pairwise disjoint and μ-measurable such that g is compactly supported and vanishes outside $\{u = -\infty\}$. Choose to each E_j, $1 \leq j \leq m$, increasing sequences $[K_j^p]_{p=1}^\infty$ of compact subsets of E_j such that $\chi_p = \sum_{j=1}^m \chi_{K_j^p}$ converges to $\sum_{j=1}^m \chi_{E_j}$ a.e. w.r.t. μ, as $p \to +\infty$. Then $\chi_p \in T$ and $g\chi_p \in T$. Furthermore, if $f_0 \in T$ with $f_0 \leq g$, then $f_0\chi_p \in T$ and $f_0\chi_p \leq g\chi_p$. Hence $u^{f_0\chi_p} \geq u^{g\chi_p}$. By the first part of the proof we have that $\left(dd^c u^{f_0\chi_p}\right)^n = f_0\chi_p(dd^c u)^n$ and $(dd^c u^{g\chi_p})^n = g\chi_p(dd^c u)^n$. Theorem 3.6 implies that $\lim_{p\to+\infty} u^{f_0\chi_p} = u^{f_0}$, hence $u^g \geq \lim_{p\to+\infty} u^{g\chi_p}$. Thus, $u^g = \lim_{p\to+\infty} u^{g\chi_p} \in \mathcal{F}$ and $(dd^c u^g)^n = g(dd^c u)^n$. □

Theorem 4.7. *Let $u \in \mathcal{E}$ and let $0 \leq g \leq 1$ be a μ_u-measurable function that vanishes outside $\{u = -\infty\}$. Then $u^g \in \mathcal{E}$ and $(dd^c u^g)^n = g(dd^c u)^n$.*

Proof. Let $[g_j]$, $g_j \in S$, be an increasing sequence that converges pointwise to g, as $j \to +\infty$. If $f \in T$ with $f \leq g$, then by Lemma 4.6 we have that $\min(f, g_j) \in S$ and $\left(dd^c u^{\min(f,g_j)}\right)^n = \min(f, g_j)(dd^c u)^n$. From Theorem 3.6 it follows that $[u^{\min(f,g_j)}]$ is a decreasing sequence that converges pointwise to u^f, as $j \to +\infty$. Thus, $u^f \geq \lim_{j\to+\infty} u^{g_j}$ for every $f \in T$ with $f \leq g$. Definition 4.5 yields that $u^g = \lim_{j\to+\infty} u^{g_j}$ and therefore it follows from Lemma 4.6 that $u^g \in \mathcal{E}$ and $(dd^c u^g)^n = g(dd^c u)^n$. □

Remark. Let u and g be as in Theorem 4.7. If $(dd^c u)^n$ vanishes on pluripolar sets, then it follows from Theorem 4.7 and the remark after Lemma 4.1 that $u^g = 0$ on Ω.

Corollary 4.8. *Let $u \in \mathcal{E}$ and f, g, $0 \leq f, g \leq 1$, be two μ_u-measurable functions which vanishes outside $\{u = -\infty\}$. If $f = g$ a.e. w.r.t. μ_u, then $u^f = u^g$.*

Proof. Let $u \in \mathcal{E}$ and assume for now that $f, g \in S$. Then by Lemma 4.6 we have that $u^f, u^g \in \mathcal{F}$, $u^f \geq u^{\max(f,g)}$ and

$$\left(dd^c u^f\right)^n = f(dd^c u)^n = \max(f,g)(dd^c u)^n = \left(dd^c u^{\max(f,g)}\right)^n .$$

Hence, by Theorem 3.6 we have that $u^f = u^{\max(f,g)}$. Similarly we get that $u^g = u^{\max(f,g)}$. Thus, $u^f = u^g$.

For the general case let $[\Omega_j]$ be a fundamental sequence and let f, g, $0 \leq f, g \leq 1$, be two μ_u-measurable functions that vanishes outside $\{u = -\infty\}$. Our assumption that $f = g$, implies that $\chi_{\Omega_j} f = \chi_{\Omega_j} g$ a.e. w.r.t. μ and by the first part of the proof we get that $u^{\chi_{\Omega_j} f} = u^{\chi_{\Omega_j} g}$. The proof is then completed by letting $j \to +\infty$. □

Example 4.9 shows that there exists a measure $g(dd^c u)^n$ carried by a pluripolar set that is not a discrete measure.

Example 4.9. Let μ be a positive measure with no atoms and with support in a compact polar subset the unit disc $\mathbb{D}$ (see e.g. [23]; p.82, and [5]; chapter IV, Theorem 1) Let u be the subharmonic Green potential of μ. Consider $\nu = \mu \times \cdots \times \mu$ (n-times) and $v(z_1, \ldots, z_n) = \max(u(z_1), ..., u(z_n))$ on $\mathbb{D} \times \cdots \times \mathbb{D}$ (n-times). Then $v \in \mathcal{F}$, $(dd^c v)^n = \nu$, ν has no atoms and it is supported by a pluripolar set.

Lemma 4.10. *Assume that α,β_1,β_2 are non-negative measures defined on Ω which satisfies the following conditions:*

(1) *α vanishes on every pluripolar subset of Ω,*
(2) *there exists a pluripolar sets $A\subset\Omega$ such that $\beta_1(\Omega\backslash A)=0$.*
(3) *for every $\rho\in\mathcal{E}_0\cap C(\bar\Omega)$ it holds that*

$$\int_\Omega(-\rho)\,\beta_1\leq\int_\Omega(-\rho)\,(\alpha+\beta_2)<+\infty\,.$$

Then we have that

$$\int_\Omega(-\rho)\,\beta_1\leq\int_\Omega(-\rho)\,\beta_2\,,$$

for every $\rho\in\mathcal{E}_0\cap C(\bar\Omega)$.

Proof. Since A is pluripolar and Ω is bounded there exists a function $\varphi\in\mathcal{PSH}(\Omega)$, $\varphi\leq 0$, such that $A\subseteq\{\varphi=-\infty\}$. Take $\rho\in\mathcal{E}_0\cap C(\bar\Omega)$ and set $\rho_j=\max\left(\rho,\frac{\varphi}{j}\right)$. Then we have that $\int_\Omega(-\rho_j)\,\beta_1\leq\int_\Omega(-\rho_j)\,(\alpha+\beta_2)<+\infty$ and by letting $j\to+\infty$ we get that

$$\int_{\{\varphi=-\infty\}}(-\rho)\,\beta_1\leq\int_{\{\varphi=-\infty\}}(-\rho)\,(\alpha+\beta_2)\,.$$

But α vanishes on pluripolar sets and β_1 and β_2 are carried by sets contained in $\{\varphi=-\infty\}$. Thus,

$$\int_\Omega(-\rho)\,\beta_1\leq\int_\Omega(-\varphi)\,\beta_2\,,$$

for every $\rho\in\mathcal{E}_0\cap C(\bar\Omega)$. □

Let $u\in\mathcal{E}$, then by Theorem 5.11 in [7] there exist functions $\phi_u\in\mathcal{E}_0$ and $f_u\in L^1_{loc}\big((dd^c\phi_u)^n\big)$, $f_u\geq 0$ such that $(dd^cu)^n=f_u\,(dd^c\phi_u)^n+\beta_u$. The non-negative measure β_u is such that there exists a pluripolar set $A\subseteq\Omega$ such that $\beta_u(\Omega\backslash A)=0$. In Lemma 4.11 we will use the notation that $\alpha_u=f_u\,(dd^c\phi_u)^n$ and β_u refereing to this decomposition.

Lemma 4.11. *Let $u,v\in\mathcal{E}$. If there exists a function $\varphi\in\mathcal{E}$ such that $(dd^c\varphi)^n$ vanishes on pluripolar sets and $|u-v|\leq-\varphi$, then $\beta_u=\beta_v$.*

Proof. Let $\Omega'\Subset\Omega$. It follows from Lemma 4.1 that there is no loss of generality to assume that $u,v,\varphi\in\mathcal{F}$, since it is sufficient to prove that $\beta_u=\beta_v$ on Ω'. The assumption that $|u-v|\leq-\varphi$ yields that $v+\varphi\leq u$ and therefore it follows from Lemma 3.3 that

$$\int_\Omega(-\rho)(dd^cu)^n\leq\int(-\rho)(dd^c(v+\varphi))^n<+\infty\,,\tag{4.1}$$

where $\rho\in\mathcal{E}_0$. Since $\sum_{j=1}^n\binom{n}{j}(dd^c\varphi)^j\wedge(dd^cv)^{n-j}\ll C_n$ we have that $\beta_{v+\varphi}=\beta_v$ and

$$\alpha_{v+\varphi}=\alpha_v+\sum_{j=1}^n\binom{n}{j}(dd^c\varphi)^j\wedge(dd^cv)^{n-j}\,.$$

Lemma 4.10 and inequality (4.1) yields that

$$\int_\Omega(-\rho)\,\beta_u\leq\int_\Omega(-\rho)\,\beta_v\,,$$

for every $\rho \in \mathcal{E}_0$. In a similar manner we get that

$$\int_\Omega (-\rho)\,\beta_v \le \int_\Omega (-\rho)\,\beta_u\,,$$

for every $\rho \in \mathcal{E}_0$. From Lemma 3.1 in [7] it now follows that $\beta_u = \beta_v$. □

Lemma 4.12. *Let $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$.*

(1) *If $v \in \mathcal{N}$, $(dd^cv)^n$ is carried by a pluripolar set, and $\int_\Omega(-\rho)(dd^cv)^n < +\infty$ for all $\rho \in \mathcal{E}_0 \cap C(\bar\Omega)$, then*

$$u = \sup\{\varphi \in \mathcal{PSH}(\Omega):\ \varphi \le \min(v,H)\} \in \mathcal{N}(H)\,,$$

and $(dd^cu)^n = (dd^cv)^n$.

(2) *Assume that $\psi \in \mathcal{N}$, $(dd^c\psi)^n$ vanishes on pluripolar sets, $v \in \mathcal{N}(H)$, $(dd^cv)^n$ is carried by a pluripolar set, and $\int_\Omega(-\rho)((dd^c\psi)^n+(dd^cv)^n) < +\infty$ for all $\rho \in \mathcal{E}_0 \cap C(\bar\Omega)$. If u is the function defined on Ω by*

$$u = \sup\left\{\varphi:\ \varphi \in \mathcal{B}\left((dd^c\psi)^n, v)\right)\right\}\,,$$

where

$$\mathcal{B}\left((dd^c\psi)^n, v\right) = \left\{\varphi \in \mathcal{E}:\ (dd^c\psi)^n \le (dd^c\varphi)^n \text{ and } \varphi \le v\right\}\,,$$

then $u \in \mathcal{N}(H)$ and $(dd^cu)^n = (dd^c\psi)^n + (dd^cv)^n$.

Proof. (1): Since $\min(v,H)$ is a negative and upper semicontinuous function we have that $u \in \mathcal{PSH}(\Omega)$ and $H \ge u \ge v + H$. Furthermore, $u \in \mathcal{N}(H)$, since $v \in \mathcal{N}(H)$. By Theorem 2.4 we can choose a decreasing sequence $[v_j]$, $v_j \in \mathcal{E}_0 \cap C(\bar\Omega)$, that converge pointwise to v as $j \to +\infty$, and use Theorem 3.7 to solve $(dd^cw_j)^n = (dd^cv_j)^n$, $w_j \in \mathcal{N}(H)$, $j \in \mathbb{N}$. Consider

$$u_j = \sup\{\varphi \in \mathcal{PSH}(\Omega):\ \varphi \le \min(v_j,H)\} \in \mathcal{E}_0(H)\,.$$

Then $u_j \ge w_j$, so by Lemma 3.3 $\int_\Omega(-\rho)(dd^cu_j)^n \le \int_D(-\rho)(dd^cw_j)^n$. Corollary 3.4 now yields that

$$\int_\Omega(-\rho)(dd^cu)^n \le \int_\Omega(-\rho)(dd^cv)^n \quad \text{for all } \rho \in \mathcal{E}_0 \cap C(\bar\Omega)\,,$$

and therefore it follows that $(dd^cu)^n$ is carried by $\{u = -\infty\}$ and since $v \ge u \ge v + H$ it follows from Lemma 4.11 that $(dd^cu)^n = (dd^cv)^n$. Thus, part (1) of this proof is completed.

(2): Using the classical Choquet's lemma (see e.g. [16]) and Proposition 4.3 in [22] we derive that $u \in \mathcal{E}$ and $(dd^cu)^n \ge (dd^c\psi)^n$. Note that $u \in \mathcal{PSH}(\Omega)$, $u \le 0$, as soon as v is only negative and upper semicontinuous and $\mathcal{B}\left((dd^c\psi)^n, v\right) \ne \emptyset$. Theorem 5.11 in [7] gives that $(dd^cu)^n = \alpha+\beta$, where α and β are positive measures defined on Ω, such that α vanishes on all pluripolar sets and β is carried by a pluripolar set. The function $(\psi + v)$ belongs to $\mathcal{B}\left((dd^c\psi)^n, v\right)$ and therefore we have that $v + \psi \le u \le v$. Hence $u \in \mathcal{N}(H)$. By Lemma 4.11 we have that $\beta = (dd^cv)^n$, and we have already noted that $\alpha \ge (dd^c\psi)^n$. Proposition 2.5 implies that there exists a decreasing sequence, $[v_j]$, $v_j \in \mathcal{E}_0(H)$, that converges pointwise to v, as $j \to +\infty$. Now,

$$\int_\Omega(-\rho)\big((dd^c\psi)^n + (dd^cv_j)^n\big) < +\infty \quad \text{for all } \rho \in \mathcal{E}_0 \cap C(\bar\Omega)$$

so by the last remark in section 3 and Theorem 3.7, there exists a unique function $w_j \in \mathcal{N}(H)$ with $(dd^c w_j)^n = (dd^c\psi)^n + (dd^c v_j)^n$. It follows from Corollary 3.2 that $w_j \in \mathcal{B}\left((dd^c\psi)^n, v_j\right)$, so if we let

$$u_j = \sup\left\{\varphi : \ \varphi \in \mathcal{B}\left((dd^c\psi)^n, v_j\right)\right\} ,$$

then $[u_j]$ decreases pointwise to u, as $j \to +\infty$. Furthermore, Lemma 3.3 implies that

$$\int_\Omega (-\rho)(dd^c u_j)^n \leq \int_\Omega (-\rho)(dd^c w_j)^n = \int_\Omega (-\rho)\big((dd^c\psi)^n + (dd^c v_j)^n\big) .$$

Let $j \to +\infty$, then Corollary 3.4 yields that

$$\int_\Omega (-\rho)(dd^c u)^n \leq \int_\Omega (-\rho)\big((dd^c\psi)^n + \beta\big) ,$$

hence $\int_\Omega(-\rho)(\alpha+\beta) \leq \int_\Omega (-\rho)\big((dd^c\psi)^n + \beta\big)$. Since we know that $\alpha \geq (dd^c\psi)^n$ it follows that for all $\rho \in \mathcal{E}_0 \cap C(\bar{\Omega})$ we have that $\int_\Omega \rho\alpha = \int_\Omega \rho (dd^c\psi)^n$, and therefore is $\alpha = (dd^c\psi)^n$. Thus, this proof is completed. □

Theorem 4.13. *Assume that μ is a non-negative measure.*

(1) *There exist functions $\phi \in \mathcal{E}_0$, $f \in L^1_{loc}\big((dd^c\phi)^n\big)$, $f \geq 0$, such that*

$$\mu = f\,(dd^c\phi)^n + \nu ,$$

where the non-negative measure ν is carried by a pluripolar subset of Ω.

(2) *If there exists a function $w \in \mathcal{E}$ with $\mu \leq (dd^c w)^n$, then there exist functions $\psi, v \in \mathcal{E}$, $v, \psi \geq w$, such that*

$$(dd^c\psi)^n = f(dd^c\phi)^n$$
$$(dd^c v)^n = \nu ,$$

where ν is carried by $\{v = -\infty\}$.

(3) *If there exists a function $w \in \mathcal{E}$ with $\mu \leq (dd^c w)^n$, then to every $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$ there exists a function $u \in \mathcal{E}$, $w + H \leq u \leq H$, with $(dd^c u)^n = \mu$. In particular, if $w \in \mathcal{N}$, then $u \in \mathcal{N}(H)$.*

Proof. (1): This is Theorem 5.11 in [7].

(2): Using the Radon-Nikodym theorem and the decomposition in part (1) (with the same notation) we obtain that

$$f(dd^c\phi)^n = \tau\chi_{\{w>-\infty\}}(dd^c w)^n \quad \text{and} \quad \nu = \tau\chi_{\{w=-\infty\}}(dd^c w)^n ,$$

where $0 \leq \tau \leq 1$ is a Borel function. For each $j \in \mathbb{N}$, let μ_j be the measure defined by $\mu_j = \min(\varphi, j)(dd^c\phi)^n$. Hence, $\mu_j \leq \left(dd^c(j^{\frac{1}{n}}\,\psi)\right)^n$ and therefore by Kołodziej's theorem (see [17]) there exists a uniquely determined function $\psi_j \in \mathcal{E}_0$ such that $(dd^c\psi_j)^n = \mu_j$. The comparison principle (Corollary 3.2) imply that $\psi_j \geq w$ and that $[\psi_j]$ is a decreasing sequence. The function $\psi = \lim_{j\to+\infty}\psi_j$ is then in $\mathcal{E}$ and $(dd^c\psi)^n = f\,(dd^c\phi)^n$. Theorem 4.7 implies that exists a functions $v \in \mathcal{E}$ such that $(dd^c v)^n = \nu$ and $v \geq w$. Thus,

$$(dd^c\psi)^n = f(dd^c\phi)^n \quad \text{and} \quad (dd^c v)^n = \nu .$$

(3): Continuing with the same notations as in part (1) and (2), we choose an increasing sequence of simple functions $[g_j]$, $\operatorname{supp} g_j \Subset \Omega$, that converges to $g = \chi_{\{w=-\infty\}}\tau$, as $j \to +\infty$. By Theorem 4.7 we have that $w^{g_j} \in \mathcal{F}$, $(dd^c w^{g_j})^n =$

$g_j(dd^cw)^n$ and $[w^{g_j}]$ is a decreasing sequence that converges pointwise to w^g, as $j\to+\infty$. Moreover $w^g\geq w$. Hence $(dd^cw^g)^n=\chi_{\{w=-\infty\}}\tau(dd^cw)^n$. Set

$$u_j=\sup\left\{\varphi\in\mathcal{B}\left((dd^c\psi_j)^n,\min(w^{g_j},H)\right)\right\}\,,$$

where

$$\mathcal{B}\left((dd^c\psi_j)^n,\min(w^{g_j},H)\right)=\left\{\varphi\in\mathcal{E}:\ (dd^c\psi_j)^n\leq(dd^c\varphi)^n \text{ and } \varphi\leq\min(w^{g_j},H)\right\}\,.$$

This construction implies that $[u_j]$ is a decreasing sequence. The sequence $[u_j]$ converges to some plurisubharmonic function u, as $j\to+\infty$, and by Lemma 4.12 $u_j\in\mathcal{N}(H)$ with $(dd^cu_j)^n=(dd^c\psi_j)^n+(dd^cw^{g_j})^n$. Furthermore, we have that that $w+H\leq u_j\leq H$. We conclude the proof by letting $j\to+\infty$.

□

Remark. Theorem 4.13 generalize Theorem 4.4 in [1], Theorem 6.2 in [7], and Corollary 1 in [28].

Remark. Let $u_1,\dots,u_n\in\mathcal{E}$. Then it follows from Theorem 4.13 that there exists a function $u\in\mathcal{E}$ such that $(dd^cu)^n=dd^cu_1\wedge\cdots\wedge dd^cu_n$.

References

[1] Åhag P. & Czyż R., On the Cegrell classes, Math. Z. 256 (2007), 243-264.

[2] Błocki Z., On the definition of the Monge-Ampère operator in $\mathbb{C}^2$, Math. Ann. 328 (2004), 415-423.

[3] Błocki Z., The domain of definition of the complex Monge-Ampère operator, Amer. J. Math. 128 (2006), 519-530.

[4] Bracci F. & Patrizio G., Monge-Ampère foliations with singularities at the boundary of strongly convex domains, Math. Ann. 332 (2005), 499-522.

[5] Carleson L., Selected problems on exceptional sets, Van Nostrand Mathematical Studies, No. 13, 1967.

[6] Cegrell U., Pluricomplex energy, Acta Math. 180 (1998), 187-217.

[7] Cegrell U., The general definition of the complex Monge-Ampère operator, Ann. Inst. Fourier (Grenoble) 54 (2004), 159-179.

[8] Cegrell U., A general Dirichlet problem for the complex Monge-Ampère operator, Ann. Polon. Math. (in press).

[9] Cegrell U., Kołodziej S. & Zeriahi A., Subextension of plurisubharmonic functions with weak singularites, Math. Z. 250 (2005), 7-22.

[10] Celik H. I. & Poletsky E. A., Fundamental solutions of the complex Monge-Ampère equation, Ann. Polon. Math. 67 (1997), 103-110.

[11] Demailly J.-P., Mesures de Monge-Ampère et mesures pluriharmoniques, Math. Z. 194 (1987), 519-564.

[12] Demailly J.-P., Monge-Ampère operators, Lelong numbers and intersection theory, Complex Analysis and Geometry, Univ. Series in Math., Plenum Press, New-York, 1993.

[13] Demailly J.-P., Complex analytic and algebraic geometry, self published e-book, (`http://www-fourier.ujf-grenoble.fr/~demailly/`).

[14] Kiselman, C.O., Sur la définition de l'opérateur de Monge-Ampère complexe, Complex analysis (Toulouse, 1983), 139-150, Lecture Notes in Math. 1094, Springer, Berlin, 1984.

[15] Klimek M., Extremal plurisubharmonic functions and invariant pseudodistances, Bull. Soc. Math. France 113 (1985), 231-240.

[16] Klimek M., Pluripotential theory, The Clarendon Press, Oxford University Press, New York, 1991.

[17] Kołodziej S., The range of the complex Monge-Ampère operator II, Indiana Univ. Math. J. 44 (1995), 765-782.

[18] Kołodziej S., The complex Monge-Ampère equation and pluripotential theory, Mem. Amer. Math. Soc. 178 (2005).

[19] Lelong P., Fonction de Green pluricomplexe et lemmes de Schwarz dans les espaces de Banach, J. Math. Pures Appl. 68 (1989), 319-347.

[20] Lempert L., La métrique de Kobayashi et la représentation des domaines sur la boule, Bull. Soc. Math. France 109 (1981), 427-474.
[21] Lempert L., Solving the degenerate complex Monge-Ampère equation with one concentrated singularity, Math. Ann. 263 (1983), 515-532.
[22] Nguyen V.K. & Pham H.H., A comparison principle for the complex Monge-Ampère operator in Cegrell's classes and applications, Trans. Amer. Math. Soc. (in press).
[23] Ransford T., Potential theory in the complex plane, London Mathematical Society Student Texts, 28. Cambridge Univ. Press, Cambridge, 1995
[24] Rashkovskii A., Relative types and extremal problems for plurisubharmonic functions, Int. Math. Res. Not. 2006, Art. ID 76283, 26 pp.
[25] Siu Y.T., Extension of meromorphic maps into Kähler manifolds, Ann. of Math. (2) 102 (1975), 421-462.
[26] Xing Y., Continuity of the complex Monge-Ampère operator, Proc. Amer. Math. Soc. 124 (1996), 457-467.
[27] Xing Y., Complex Monge-Ampère equations with a countable number of singular points, Indiana Univ. Math. J. 48 (1999), 749-765.
[28] Xing Y., A decomposition of complex Monge-Ampère measures, Ann. Polon. Math. 92 (2007), 191-195.
[29] Zahariuta V.P., Spaces of analytic functions and maximal plurisubharmonic functions, Doc. Sci. Thesis, 1984.
[30] Zeriahi A., Pluricomplex Green functions and the Dirichlet problem for the complex Monge-Ampère operator, Michigan Math. J. 44 (1997), 579-596.

Department of Mathematics, Umeå University, SE-901 87 Umeå, Sweden

Department of Mathematics, Umeå University, SE-901 87 Umeå, Sweden
E-mail address: Urban.Cegrell@math.umu.se

Institute of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland
E-mail address: Rafal.Czyz@im.uj.edu.pl

Department of Mathematics, Truong Dai hoc Su pham Ha Noi, 136 Xuan Thuy, Cau Giay, Ha Noi, Vietnam
E-mail address: phhiep_vn@yahoo.com